



Critical exponent for the quantum Hall transition

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(Received 13 May 2009; published 9 July 2009)

We report an estimate $\nu=2.593$ [2.587,2.598] of the critical exponent of the Chalker-Coddington model of the integer quantum Hall effect that is significantly larger than previous numerical estimates and in disagreement with experiment. This suggests that models of noninteracting electrons cannot explain the critical phenomena of the integer quantum Hall effect.

DOI: [10.1103/PhysRevB.80.041304](https://doi.org/10.1103/PhysRevB.80.041304)

PACS number(s): 73.43.-f, 72.15.Rn

The most important characteristics of a continuous phase transition are its critical exponents. The critical exponents are a quantitative characteristic of the critical fixed point. Agreement of theory with experiment implies that the critical fixed point studied theoretically is the same fixed point observed in the experiment. For the integer quantum Hall transition¹ the value of the critical exponent ν that describes the divergence of the localization length ξ of the electrons has now been measured to an accuracy of a few percent,^{2,3} so comparison with experiment has become a stringent test of theory. One of the most surprising aspects of the integer quantum Hall transition is the agreement between experiment and theory⁴⁻⁷ concerning the value of the critical exponent. The agreement is surprising because the Coulomb interaction between the electrons is not included in the theoretical models.⁸⁻¹⁰ It suggests that the Coulomb interaction is not relevant at the integer quantum Hall critical point. Yet, this is immediately contradicted when we look at the dynamical critical exponent z . For models of noninteracting electrons z is exactly 2,¹¹ whereas the measured value is ≈ 1 .³

We show in this Rapid Communication that previous theoretical work has significantly underestimated the critical exponent ν for noninteracting electrons and that there is in fact a clear disagreement with experiment. This suggests that models of noninteracting electrons cannot explain the critical phenomena of the integer quantum Hall effect.

Noninteracting electron models of the integer quantum Hall transition are also of interest in their own right. In particular, though the critical-field theory for such models is as yet unknown,¹² it has been speculated that it should have conformal invariance.¹³ If so, a scaling relation follows,¹⁴ which we test by comparing recent multifractal analyses^{15,16} with our work.

The integer quantum Hall effect¹ occurs in two-dimensional (2D) electron gases that are subject to a large perpendicular magnetic field. The application of the field results in the quantization of the kinetic energy of the electrons and the formation of Landau levels. Impurity scattering causes the Landau levels to broaden into Landau bands. The states at the center of the Landau band are critical, while other states are localized. A quantum Hall transition between quantized values of the Hall resistance occurs whenever the

Fermi energy passes through the center of a Landau band.^{6,17} The critical exponent ν describes the divergence of the localization length,

$$\xi \sim |x - x_c|^{-\nu},$$

near the critical energy. Here, x can be any control parameter, for example, the Fermi energy, that drives the two-dimensional electron gas through the transition at x_c . In common with other continuous phase transitions, the value of the critical exponent is expected to exhibit a high degree of universality.¹³

We have performed a finite-size scaling (FSS) analysis of the quantum Hall transition in the Chalker-Coddington network model.^{10,18} In this model the motion of the electron in a random potential and quantizing magnetic field is replaced by the transmission of an electron through a network of links and nodes. The links describe electron motion along lines of constant potential, and the nodes describe the scattering of electrons at saddle points of the potential. The Coulomb interaction between the electrons is ignored.

We use the transfer-matrix method to estimate the smallest positive Lyapunov exponent of a very long network consisting of L layers. Each layer consists of two sublayers: one a transverse array of N nodes of type A and the other a transverse array of N nodes of type B. A transfer matrix T_l of size $2N \times 2N$ relates the N right-going and N left-going flux amplitudes at the left of the layer to the similar quantities at the right of the layer,

$$\begin{pmatrix} a'_1 \\ b'_1 \\ \vdots \\ a'_N \\ b'_N \end{pmatrix} = T_l \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_N \\ b_N \end{pmatrix}.$$

The explicit form of the transfer matrix is

$$T_l = BV_lAU_l.$$

The scattering at nodes of type A is described by the matrix

$$A = \begin{bmatrix} 1/t & r/t & & & \\ r/t & 1/t & & & \\ & & 0_2 & \cdots & 0_2 \\ & 0_2 & 1/t & r/t & \\ & \vdots & r/t & 1/t & \ddots \\ & & \ddots & \ddots & \ddots \\ 0_2 & & \cdots & 0_2 & 1/t & r/t \\ & & & r/t & 1/t & \end{bmatrix}.$$

Here, 0_2 is the 2×2 zero matrix, t is the probability amplitude with which current from the left is transmitted to the right, while r is the amplitude with which current from the left is reflected back to the left. Nodes of type B are obtained from nodes of type A by a rotation of 90° . Imposing periodic boundary condition in the transverse direction we then have

$$B = \begin{bmatrix} 1/r & 0 & 0 & \cdots & \cdots & t/r \\ 0 & 1/r & t/r & & & 0 \\ 0 & t/r & 1/r & & 0_2 & \cdots \\ \vdots & & & 0_2 & \ddots & \ddots \\ 0 & \vdots & & \ddots & 1/r & t/r \\ 0 & & \vdots & & t/r & 1/r \\ t/r & 0 & 0 & \cdots & 0 & 0 & 1/r \end{bmatrix}.$$

The transmission and reflection amplitudes are parameterized by the parameter x according to

$$t = [\exp(2x) + 1]^{-1/2}, \quad r = \sqrt{1 - t^2}.$$

The parameter x may be interpreted as the energy of the electron measured from the Landau band center scaled by the Landau band width. The matrix U_l is a diagonal matrix with elements

$$(U_l)_{m,n} = \delta_{m,n} \exp(i\varphi_{l,m}),$$

and similarly V_l is a diagonal matrix with elements

$$(V_l)_{m,n} = \delta_{m,n} \exp(i\varphi'_{l,m}).$$

Since the distances between the nodes are random, we suppose that the phases are independently and uniformly distributed on $[0, 2\pi)$. (These random numbers were generated using the routine ran described in Chapter B7 of Ref. 19.)

The transfer matrix of the network model is equal to the product of the transfer matrices of the layers

$$T = \prod_{l=1}^L T_l.$$

From this matrix we can define a Hermitian matrix Ω by

$$\Omega = \ln(T^\dagger T).$$

As a consequence of the conservation of flux the eigenvalues of this matrix occur in pairs of opposites sign

$$\{+\nu_1, \dots, +\nu_N, -\nu_N, \dots, -\nu_1\}, \quad \nu_1 > \dots > \nu_N > 0.$$

We are interested in the smallest positive Lyapunov exponents, which is obtained in the following limiting procedure:

TABLE I. The least-squares fit of the FSS model to the simulation data. The number of data points is $N_D=217$ and the number of parameters is $N_P=9$. The minimum value of $\chi^2=199.8$ and the goodness of fit $p=0.6$. The series for F_0 , u_0 , F_1 , and u_1 was truncated at orders 6, 3, 0, and 2, respectively.

	Estimate	95% confidence interval
α	0.3857	[0.3849, 0.3866]
Γ_c	0.780	[0.767, 0.788]
y	-0.17	[-0.21, -0.14]

$$\gamma = \lim_{L \rightarrow \infty} \frac{\nu_N}{2L}.$$

The value obtained is independent of the particular sequence of transfer matrices in the transfer-matrix product for almost all sequences. Truncation of the transfer-matrix multiplication at a large but finite L yields an estimate of the Lyapunov exponent with a known precision. Repeated QR factorizations are needed to avoid a loss of precision due to round-off error.²⁰

To analyze the simulation data we assume that the dimensionless quantity,

$$\Gamma = \gamma N,$$

obeys an FSS law

$$\Gamma = F_0(N^\alpha(x - x_c)).$$

Here, α is the reciprocal of the critical exponent ν . In practice, the data deviate from this law. This is taken into account by including corrections to scaling that arise from nonlinearity of the scaling variables and irrelevant scaling variables. In previous work²¹ on the three-dimensional Anderson model, we have found that the fitting formula

$$\Gamma = F_0(N^\alpha u_0) + F_1(N^\alpha u_0) N^y u_1, \quad (1)$$

works well. Here, $u_0 \equiv u_0(x)$ and $u_1 \equiv u_1(x)$ are, respectively, relevant and irrelevant scaling variables, and F_0 and F_1 are corresponding scaling functions. The exponent y associated with the irrelevant variable is negative, $y < 0$. The relevant variable is zero at the critical point. For the Chalker-Coddington model the critical point is known to be exactly $x_c=0$ and does not need to be found by fitting the numerical data, so $u_0(0)=0$.

When periodic boundary conditions are imposed, swapping t and r and translating the network so that the A and B nodes are interchanged yield an identical network. As a result of this symmetry Γ is an even function of x . (Note that this holds only for periodic boundary conditions.) To reflect this, we make F_0 , F_1 , and u_1 even, and u_0 odd, functions. The condition on u_0 follows because the relevant variable is zero at the critical point.

We are able to fit our numerical data to Eq. (1) provided that data for smaller system sizes ($N < 16$) are excluded. All the functions appearing in Eq. (1) are expanded in Taylor series. The results of the fit are summarized in Table I. The

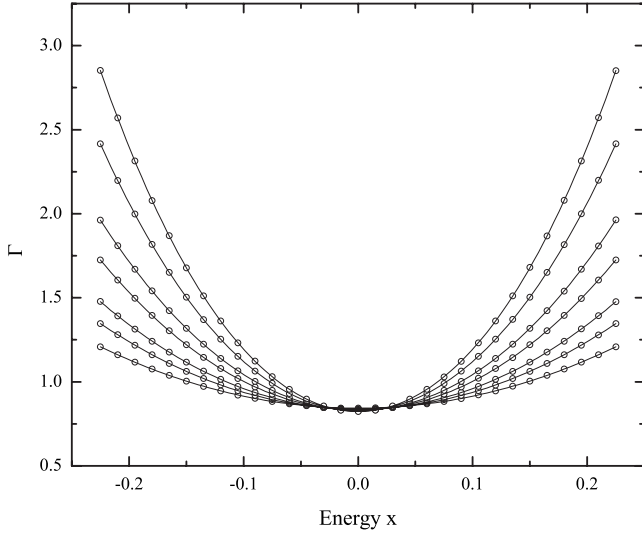


FIG. 1. Simulation data and FSS fit for the Chalker-Coddington model. The different lines correspond to different numbers of nodes $N=16, 24, 32, 48, 64, 96, 128$.

data and the fit are plotted in Figs. 1 and 2. The confidence intervals have been obtained from Monte Carlo simulation.²²

It is clear from Fig. 2 that corrections to scaling are on the order of a few percent for the smallest N . The precision of our data is 0.03%, which requires $L \approx 10^8 \sim 10^9$ transfer-matrix multiplications to achieve. Such high precision is required because the dependence on energy of Γ is very weak near the critical point for the available N and the critical exponent is, in effect, estimated from the variation in the curvature with N .

We have tested the stability of this fit when the data are filtered versus the range of Γ (Table II), the range of N (Table III), and also checked for stability against increases on the order of the various Taylor expansions (Table IV). In all cases there is a large overlap of the confidence intervals with the fit in Table I.

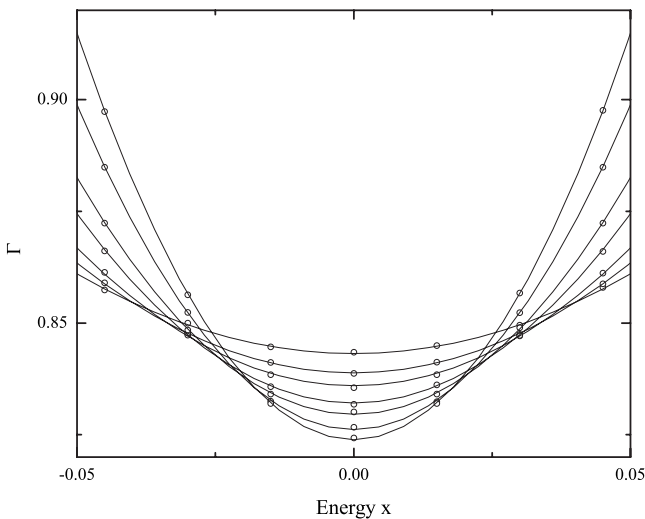


FIG. 2. The same as Fig. 1 but with the focus on data near the critical point so as to make the existence of corrections to scaling due to an irrelevant scaling variable evident.

TABLE II. The best fit of α and Γ_c with 95% confidence intervals. The data have been filtered using the condition on Γ at the left.

$\Gamma < 2.0$	0.3858	[0.3850, 0.3867]	0.779	[0.767, 0.788]
$\Gamma < 1.5$	0.3855	[0.3847, 0.3864]	0.781	[0.769, 0.790]
$\Gamma < 1.0$	0.3826	[0.3796, 0.3858]	0.789	[0.778, 0.797]

Our FSS analysis yields the following best fit value and 95% confidence interval for the critical exponent:

$$\nu = 2.593 \quad [2.587, 2.598]. \quad (2)$$

This result is consistent with but also a considerable improvement on the original estimate $\nu = 2.5 \pm .5$ of Chalker and Coddington.¹⁰

Our estimate is significantly larger than the oft quoted result of $\nu = 2.34 \pm .04$ of Huckestein and Kramer⁵ for a random Landau matrix model. This disagreement could be taken as evidence against the universality of the critical exponent. However, we feel that a more likely explanation is that the precision claimed by Huckestein and Kramer⁵ is too optimistic. This may be clarified in future work.

Our result is also different from the analytical result $\nu = 7/3$ of Mil'nikov and Sokolov.⁴ However, this value is not expected to describe the true critical point but an intermediate behavior not too close to the critical point.

Our result also disagrees with $\nu = 2.37 \pm 0.02$ of Cain *et al.*⁷ based on a real-space renormalization-group approach.²³ A more recent calculation²⁴ on a network with a triangular lattice gave $\nu \approx 2.3 \sim 2.76$. Thus, the precision claimed by Cain *et al.*⁷ may be questionable. Moreover, this approach involves an uncontrolled approximation and so the discrepancy may not be significant.

More important than the discrepancy with previous theory is the disagreement of our estimate with recent experiments. Li *et al.*^{2,3} measured $\nu = 2.38 \pm .06$ in experiments on GaAs-AlGaAs heterostructures. In these experiments, an exponent $\kappa = 1/\nu z$ was measured. Extraction of the critical exponent ν requires an independent measurement of the dynamical exponent z . Li *et al.*^{2,3} measured $z \approx 1$ but seem to have obtained their result for ν by supposing $z = 1$. An unambiguous comparison with theory will require a more careful consideration of the precision of the estimate of z .

In our opinion, the coincidence of previous theoretical and experimental estimates of ν is not significant. Noninteracting theory predicts $z = 2$ in clear disagreement with experiment. Calculations¹¹ within the Hartree-Fock approximation suggest that the observed value of $z \approx 1$ may be explained by including the Coulomb interaction. We speculate that the discrepancy in the value of ν that we report here may also be explained in this way.

Assuming that the critical-field theory of the integer quan-

TABLE III. The best fit of α and Γ_c with 95% confidence intervals. The data have been filtered using the condition on N at the left.

$N \geq 24$	0.3854	[0.3845, 0.3864]	0.782	[0.763, 0.794]
$N \geq 32$	0.3849	[0.3837, 0.3861]	0.787	[0.768, 0.800]

TABLE IV. The best fit of α and Γ_c with 95% confidence intervals. The order of the expansion of the indicated function has been increased by two compared with Table I.

F_0	0.3858	[0.3849,0.3866]	0.779	[0.767,0.788]
u_0	0.3853	[0.3843,0.3864]	0.780	[0.767,0.789]
F_1	0.3828	[0.3754,0.3917]	0.780	[0.767,0.789]
u_1	0.3857	[0.3849,0.3865]	0.780	[0.767,0.788]

tum Hall effect has conformal invariance, it has been shown that Γ_c and α_0 (determined in multifractal analysis^{15,16}) are related by¹⁴

$$\Gamma_c = \pi(\alpha_0 - 2). \quad (3)$$

Equation (3) follows from a conformal mapping between the strip used in transfer-matrix calculations and the 2D plane used in multifractal analysis. Inserting our result for Γ_c into this formula we obtain

$$\alpha_0 = 2.248 \quad [2.244, 2.251].$$

This value is not consistent with either 2.2617 ± 0.0006 reported in Ref. 15 or 2.2596 ± 0.0004 in Ref. 16. However, we cannot conclude that the critical theory does not have conformal invariance because the leading correction to scaling decays very slowly with N , as is indicated by the small value of the irrelevant exponent y . This does not affect the estimation of the critical exponent ν but it does complicate the estimation of Γ_c and, probably, α_0 .

In conclusion, we report an estimate of the critical exponent ν of the integer quantum Hall effect that is significantly larger than both previous theoretical estimates and, more importantly, experimentally measured values. We speculate that models of noninteracting electrons cannot explain the critical phenomena of the integer quantum Hall effect. Further work is needed to come to a definite conclusion concerning the conformal invariance of the critical theory of noninteracting electron models of the integer quantum Hall transition.

This work was supported by Grant-in-Aid (Grant No. 18540382). We would like to thank H. Obuse, A. Furusaki, F. Evers, and A. M. Tsvelik for helpful comments.

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